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Dynamical cluster properties in the quantum statistical mechanics of phase transitions

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Abstract. In this paper we will investigate the influence of spontaneous symmetry breaking on the dynamical cluster properties of correlation functions of a quantum statistical ensemble, that is clustering with respect to time and space–time. While the impact of phase transitions on pure space-like clustering, i.e. the static results, is well known, less seems to be known about the dynamical behaviour of response functions and susceptibilities. We are able to make precise statements which show that, contrary to the static case, the decay properties depend sensitively on the type of Goldstone excitation.

1. Introduction

In this paper we will deal with systems which can exhibit a phase transition accompanied by the spontaneous breaking of a continuous symmetry (henceforth denoted by SSB).

It is well known that the occurrence of SSB, namely of the so called Goldstone excitations, affects the equal-time cluster properties of certain important correlation functions which are closely related to the generalised susceptibilities of the system under discussion (see, e.g., Wagner 1966, Hohenberg 1966). This interrelation between SSB and the cluster properties of correlation functions ensures that SSB means a phase transition in the classical sense, namely singularities in certain thermodynamic functions.

These equal-time cluster properties are purely static results and it seems worthwhile to study the dynamical implications of the emergence of Goldstone particles, i.e. the cluster properties in the whole four-dimensional region of space and time.

It will be shown in the following that it is possible to characterise in a relatively rigorous manner the cluster properties of the relevant correlation functions with respect to time and space–time. This is satisfying since the change in the dynamical behaviour of a system under a phase transition is of practical importance, and to make relatively precise statements usually appears to be quite difficult.

We will make only a few general assumptions which can be easily controlled in the models under discussion and which allow us to apply our approach to almost every model of a phase transition with SSB, thereby obtaining detailed information about the cluster behaviour.

One of our main assumptions will be that the Goldstone quasiparticle is relatively long-lived, in other words that the collective excitation is sufficiently sharply peaked.

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This restricts our approach to the region well below the critical point. In the region where the system behaves nearly critically the renormalisation group technique must be applied.

2. General results about the behaviour of systems in the presence of SSB

In this section we will compile some general statements and concepts needed in connection with SSB.

In the following we consider only bulk effects of the system. We therefore have to work in an infinitely extended medium, that is we have to perform the thermodynamic limit $V\to\infty$. By the usual procedure of quasi-averages, we arrive at the pure phases of the system. The thermodynamic state representing one of these phases is denoted by ω. Let $\mathscr{A}$ denote the class of quasilocal objects of the theory, which consists roughly of the measurements and operations which can be performed in a finite region of space, such as, e.g., the local densities and currents. Elements of $\mathscr{A}$ are denoted by $A, B \ldots$.

$A(\boldsymbol{x}, t)$ means the space–time translated element A and $\omega(A)$ is the expectation with respect to ω. Since ω is the limit of Gibbs states we get $\omega(A(t))=\omega(A)$; furthermore, ω is assumed to be translationally invariant. ω must fulfil the so called Kubo–Martin–Schwinger condition (KMS), which we need only in its Fourier-transformed form (see, e.g., Haag *et al* 1967).

Let $J(\boldsymbol{k}, \omega)$ denote the Fourier transform (FT) of the truncated two-point function $\langle A(\boldsymbol{x}, t)B\rangle^{\mathrm{T}} := \omega(A(\boldsymbol{x}, t)B)-\omega(A)\omega(B)$. Then the KMS condition reads

$$\text{FT of } \omega([A(\boldsymbol{x}, t), B]) = (1-\mathrm{e}^{-\beta\omega})J(\boldsymbol{k}, \omega) \tag{1}$$

with $\beta=(k_{\mathrm{B}}T)^{-1}$.

Now we assume a one-parameter symmetry group which acts on the elements of $\mathscr{A}$ to be present in the system. At least in a formal manner, an infinitesimal generator Q is given; this is usually built from a conserved density $q(\boldsymbol{x}, t)$, $\partial_0 q(\boldsymbol{x}, t)+\nabla j(\boldsymbol{x}, t)=0$:

$$Q(t) := \int \mathrm{d}\boldsymbol{x}\, q(\boldsymbol{x}, t) \qquad Q(t)=Q$$

when the symmetry is conserved. When the symmetry is spontaneously broken the above Q does not exist in the usual sense, since the equilibrium state ω appears not to be invariant under the symmetry, that is there exist elements A with $\omega([Q, A])\neq 0$.

What is well defined as an operator in the theory is $q(\boldsymbol{x}, t)$ integrated over a finite volume. To avoid artificial singularities in the theory by using a sharp volume boundary and to perform the FT more easily, we define a smooth cut-off by defining a class of functions which can be differentiated infinitely often:

$$f_R(\boldsymbol{x}) := \begin{cases} 1 & \text{for } |\boldsymbol{x}|\leqslant R \\ 0 & \text{for } |\boldsymbol{x}|>2R \end{cases} \qquad Q_R(t) := \int \mathrm{d}\boldsymbol{x}\, q(\boldsymbol{x}, t) f_R(\boldsymbol{x}).$$

Then a rigorous formulation of SSB is

$$\lim_{R\to\infty} \omega([Q_R(t), A]) \neq 0 \tag{2}$$

for at least one $A\in\mathscr{A}$. Furthermore, the limit is to be independent of t since the symmetry group commutes with the time evolution.

We restrict ourselves to short-range interactions. When the interaction is of long range new phenomena can occur and the Goldstone picture need not work in the usual manner, as, for example, in some models of superconductivity.

Before we derive the main results of this section, some words about the general properties of the two-point functions $\omega(q(\boldsymbol{x}, t)A)^{\mathrm{T}}$ and $J(\boldsymbol{k}, \omega)$ are appropriate.

Lemma 1. If $q(\boldsymbol{x}, t)$ is a density covariant under translations, in other words if $U(\boldsymbol{a}, a_0)q(\boldsymbol{x}, t)U^{-1}(\boldsymbol{a}, a_0) = q(\boldsymbol{x}+\boldsymbol{a}, t+a_0)$ with $U(\boldsymbol{a}, a_0)$ the translation operator, it can be shown that $J(\boldsymbol{k}, \omega)$ is a measure.

Proof. The proof relies on the functional analytic approach to measure theory (see, e.g., Bourbaki (1967) or Schwartz (1966)). A distribution $J(\boldsymbol{k}, \omega)$ is a measure if

$$\lim_{n\to\infty} \int J(\boldsymbol{k}, \omega) f_n(k, \omega)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega = 0 \tag{3}$$

holds for every sequence of functions which are elements of $\mathscr{C}_{00}$ (the continuous functions of compact support) with $f_n \to 0$ in the supremum-norm topology, f_n having their support in an arbitrary but fixed bounded domain of $\mathbb{R}^4$.

Several proofs are available. The most general one runs as follows. Because of the properties of $\langle q(\boldsymbol{x}, t)A(\boldsymbol{x}', t')\rangle$ and translation covariance, the Cauchy–Schwartz inequality can be applied yielding

$$\left|\int F_{qA}(\boldsymbol{x}-\boldsymbol{x}', t-t')f(x)g(x)\, \mathrm{d}x\, \mathrm{d}x'\right|^2$$
$$\leqslant \int F_{qq}(x-x')f(x)f(x')\, \mathrm{d}x\, \mathrm{d}x' \int F_{AA}(x-x')g(x)g(x')\, \mathrm{d}x\, \mathrm{d}x'$$

with $x, x' \in \mathbb{R}^4$, f and g real test functions and $F_{qA}(x-x') := \langle q(x)A(x')\rangle$.

F_{qq} and F_{AA} are positive semidefinite functions; hence their Fourier transforms $\tilde{F}_{qq}$ and $\tilde{F}_{AA}$ are positive measures by the Bochner–Schwartz theorem (see, e.g. Schwartz 1966). The Fourier-transformed version of the above inequality reads

$$\left|\int \tilde{F}_{qA}(p)\tilde{f}(p)\tilde{g}(p)\, \mathrm{d}p\right|^2 \leqslant \int \tilde{F}_{qq}(p)\tilde{f}^2(p)\, \mathrm{d}p \int \tilde{F}_{AA}(p)\tilde{g}^2(p)\, \mathrm{d}p.$$

We choose $\tilde{f}$ and $\tilde{g}$ of compact support with $\tilde{g} \equiv 1$ on the support of $\tilde{f}$. This yields

$$\left|\int \tilde{F}_{qA}(p)\tilde{f}(p)\, \mathrm{d}p\right|^2 \leqslant \int \tilde{F}_{qq}(p)\tilde{f}^2(p)\, \mathrm{d}p \times \text{constant}.$$

Choosing now a sequence $\tilde{f}_n(p)$ as in (3), the right-hand side goes to zero according to the measure property of $\tilde{F}_{qq}$; hence the left-hand side and $\tilde{F}_{qA}$ are shown to be a measure also.

We can make the following remarks.

(i) The physical implications of $J(k, \omega)$ being a measure will become clear in the following. This property is mainly a statement about the possible singularities of $J(k, \omega)$.

(ii) If $q(\boldsymbol{x}, t)$ is not translationally covariant, $J(\boldsymbol{k}, \omega)$ will have stronger singularities. For example, the densities which generate the Galilei boosts have the form $-x_i\rho(\boldsymbol{x}, t)+$

$tj_l(\boldsymbol{x}, t)$ where ρ and j denote particle density and particle current. $J(\boldsymbol{k}, \omega)$ is now the derivative of a measure and is therefore more singular.

The well known argument connecting SSB with a Goldstone excitation mode runs as follows:

$$\text{constant} = \lim_{R\to\infty} \omega([Q_R(t), A]) = \lim_R \int f_R(\boldsymbol{x})\omega([q(\boldsymbol{x}, t)A])\, d\boldsymbol{x} \tag{4}$$

$$= \lim_R \int \tilde{f}_R(\boldsymbol{k})\, e^{i\omega t}(1 - e^{-\beta\omega})J(\boldsymbol{k}, \omega)\, d\omega\, d\boldsymbol{k}$$

where a tilde denotes the Fourier transform.

Because of the definition of $f_R(\boldsymbol{x}), \{f_R(\boldsymbol{k})\}$ converges towards $\delta(\boldsymbol{k})$ in the limit $R \to \infty$. The limit is time independent, which means that in the limit $R \to \infty$ the only contribution comes from a branch going through $(\boldsymbol{k}, \omega) = (\boldsymbol{0}, 0)$. More rigorously,

$$\text{constant} \times \delta(\omega) = \lim_R \int \tilde{f}_R(\boldsymbol{k})(1 - e^{-\beta\omega})J(\boldsymbol{k}, \omega)\, d\boldsymbol{k}. \tag{5}$$

Since the left-hand side is non-zero, $J(\boldsymbol{k}, \omega)$ has to become singular in $(\boldsymbol{k}, \omega) = (\boldsymbol{0}, 0)$ in order to compensate for the vanishing of $(1 - e^{-\beta\omega})$. The precise behaviour of $J(\boldsymbol{k}, \omega)$ in the vicinity of $(\boldsymbol{k}, \omega) = (\boldsymbol{0}, 0)$ will be the main result of the rest of this section.

Without restriction we can choose $A = A^*$ (by passing from A to $\frac{1}{2}(A + A^*)$, $\frac{1}{2}i(A - A^*)$ respectively). Then the KMS condition yields

$$J(\boldsymbol{k}, \omega) - \bar{J}(-\boldsymbol{k}, -\omega) = (1 - e^{-\beta\omega})J(\boldsymbol{k}, \omega).$$

That is,

$$\begin{aligned} \operatorname{Re} J(-\boldsymbol{k}, -\omega) &= e^{-\beta\omega} \operatorname{Re} J(\boldsymbol{k}, \omega) \\ \operatorname{Im} J(-\boldsymbol{k}, -\omega) &= -e^{-\beta\omega} \operatorname{Im} (\boldsymbol{k}, \omega). \end{aligned} \tag{6}$$

Without restriction we can choose $f_R(\boldsymbol{x})$ to be real and symmetric. Then $f_R(\boldsymbol{k})$ is symmetric and we obtain

$$\begin{aligned} &\omega([Q_R(0), A]) \\ &= \int \tilde{f}_R(\boldsymbol{k})(1 - e^{-\beta\omega})J(\boldsymbol{k}, \omega)\, d\boldsymbol{k}\, d\omega \\ &= \int_{\omega \geqslant 0} (\ldots)\, d\boldsymbol{k}\, d\omega + \int_{\omega < 0} (\ldots)\, d\boldsymbol{k}\, d\omega \\ &= i \int_{\omega \geqslant 0} \tilde{f}_R(\boldsymbol{k})[(1 - e^{-\beta\omega}) - (1 - e^{\beta\omega})\, e^{-\beta\omega}] \operatorname{Im} J(\boldsymbol{k}, \omega)\, d\boldsymbol{k}\, d\omega \\ &\quad + \int_{\omega \geqslant 0} \tilde{f}_R(\boldsymbol{k})[(1 - e^{-\beta\omega}) + (1 - e^{\beta\omega})\, e^{-\beta\omega}] \operatorname{Re} J(\boldsymbol{k}, \omega)\, d\boldsymbol{k}\, d\omega. \end{aligned}$$

In the second term the expression in square brackets vanishes identically. So we arrive at the following conclusion.

Lemma 2. SSB is connected only with the imaginary part of the measure $J(\boldsymbol{k}, \omega)$. It is sufficient to take only the part $\operatorname{Im} J(\boldsymbol{k}, \omega)$ with $\omega \geqslant 0$ into account.

The first term simplifies to

$$2\mathrm{i}\int_{\omega\geq 0}\tilde{f}_R(\boldsymbol{k})(1-\mathrm{e}^{-\beta\omega})\,\mathrm{Im}\,J(\boldsymbol{k},\omega)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega.$$

Hence the singularity which shows up in $J(\boldsymbol{k},\omega)$ when SSB occurs has to be a contribution from $\mathrm{Im}\,J(\boldsymbol{k},\omega)$.

Our next aim is to analyse the singular behaviour of $J(\boldsymbol{k},\omega)$ in $(\mathbf{0},0)$ which has already been indicated in (4). According to a general theorem, we can split the measure $J(\boldsymbol{k},\omega)$ into a pure point measure J_{pp}, a singular continuous part J_{sc} and an absolutely continuous term J_{ac}:

$$J=J_{\mathrm{pp}}+J_{\mathrm{sc}}+J_{\mathrm{ac}}. \tag{7}$$

J_{pp} is a sum over discrete δ contributions in $\mathbb{R}^4$, i.e. $J_{\mathrm{pp}}(\boldsymbol{k},\omega)=\Sigma_i c_i\delta(\boldsymbol{k}-\boldsymbol{k}_i)\delta(\omega-\omega_i)$, J_{ac} is absolutely continuous with respect to Lebesgue measure, $J_{\mathrm{ac}}(\boldsymbol{k},\omega)=\hat{J}(\boldsymbol{k},\omega)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\omega$ with $\hat{J}$ a locally integrable function, and J_{sc} is a measure concentrated on a subset with Lebesgue measure zero but without discrete points.

Since we are interested only in the contribution which is responsible for the phase transition we write J as $J_{\mathrm{s}}+J_{\mathrm{n}}$ with J_{n} the part which does not contribute to $\lim_R\langle[Q_R,A]\rangle$, i.e. the contribution extrapolated from the regime where the system is normal, that is, where there is no SSB. The effect of the symmetry breaking is assumed to be concentrated completely in J_{s}. In the following theorem we state some necessary properties J_{s} has to fulfil in order that a phase transition can take place.

Theorem 1. Assuming SSB the following hold.
(i) J_{s} does not consist of a pure point contribution $c\delta(\boldsymbol{k})\delta(\omega)$.
(ii) When one can perform the limit $R\to\infty$ under the integral there is no SSB.
(iii) $J_{\mathrm{s}}(\boldsymbol{k},\omega)$ is not a continuous bounded function in $(\mathbf{0},0)$, that is, $J_{\mathrm{s}}(\boldsymbol{k},\omega)$ has to become singular in $(\mathbf{0},0)$; in particular, $J_{\mathrm{s}}(\mathbf{0},\omega)=0$ for $\omega\neq 0$.

Proof.
(i) A pure point contribution implies $\lim_{(\boldsymbol{x},t)\to\infty}\langle q(\boldsymbol{x},t)A\rangle^{\mathrm{T}}\neq 0$ but in a pure phase we have ergodicity, i.e. clustering of truncated correlations, $\lim_{(\boldsymbol{x},t)\to\infty}\langle q(\boldsymbol{x},t)A\rangle^{\mathrm{T}}=0$ (see, e.g., Ruelle 1969).

(ii) This would yield constant $\times\,\delta(\omega)=(1-\mathrm{e}^{-\beta\omega})J_{\mathrm{s}}(\mathbf{0},\omega)$. As a restriction of $J_{\mathrm{s}}(\boldsymbol{k},\omega)$ $J_{\mathrm{s}}(\mathbf{0},\omega)$ is also a measure. The left-hand side forces it to be a pure point measure in $\omega=0$; hence $J_{\mathrm{s}}(\mathbf{0},\omega)=c\delta(\omega)$. But $(1-\mathrm{e}^{-\beta\omega})$ is zero in $\omega=0$; hence the constant is zero, which means that there is no SSB at all.

(iii) With $J_{\mathrm{s}}(\boldsymbol{k},\omega)$ continuous in $(\mathbf{0},0)$ the performance of $\lim_R$ under the integral would be allowed (remember that $\tilde{f}_R$ is a δ sequence); hence (ii) follows.

To develop a feeling for what can happen we now show that both a singular continuous term and an absolutely continuous term can produce SSB. A typical contribution belonging to J_{sc} is a sharp excitation branch defined by a continuous positive function $\sigma(\boldsymbol{k})$ with $\sigma(\mathbf{0})=0$. With the symmetry condition (6) $J_{\mathrm{s}}(\boldsymbol{k},\omega)$ can be written in this case as

$$J_{\mathrm{s}}(\boldsymbol{k},\omega)=J(\boldsymbol{k})\delta(\omega-\sigma(\boldsymbol{k}))-\mathrm{e}^{-\beta|\omega|}J(-\boldsymbol{k})\delta(\omega+\sigma(\boldsymbol{k})). \tag{8}$$

$J(\boldsymbol{k})$ is to be a local integrable function; the second contribution is the quasi-hole excitation branch for $\omega < 0$. Inserting this into expression (4) we arrive at a condition $J(\boldsymbol{k})$ has to fulfil in order that a phase transition can emerge:

$$\text{constant} = 2\mathrm{i} \lim_{R} \int \tilde{f}_R(\boldsymbol{k})(1 - \mathrm{e}^{-\beta\sigma(\boldsymbol{k})})\, \mathrm{e}^{\mathrm{i}t\sigma(\boldsymbol{k})} J(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}.$$

With $\tilde{f}_R(\boldsymbol{k}) \to \delta(\boldsymbol{k})$ we get

$$J(\boldsymbol{k}) = c(\boldsymbol{k})(1 - \mathrm{e}^{-\beta\sigma(\boldsymbol{k})})^{-1} \tag{9}$$

with $c(\boldsymbol{k})$ a bounded function in $\boldsymbol{k} = \boldsymbol{0}$ and $c(\boldsymbol{0}) = -\frac{1}{2}\mathrm{i} \times \text{constant}$.

In physical terms $\delta(\omega - \sigma(\boldsymbol{k}))$ has to be interpreted as a Goldstone excitation of infinite lifetime, $\delta(\omega + \sigma(\boldsymbol{k}))$ as the corresponding hole excitation and $J(\boldsymbol{k})$ as the spectral weight of the branch. Hence we arrive at the following theorem.

Theorem 2. A sharp Goldstone excitation with dispersion law $\omega = \sigma(\boldsymbol{k})$ forces the corresponding spectral weight to become singular in $\boldsymbol{k} = \boldsymbol{0}$ like $(1 - \mathrm{e}^{-\beta\sigma(\boldsymbol{k})})^{-1}$, that is roughly as $\sigma(\boldsymbol{k})^{-1}$.

In the appendix we will discuss several realistic models which will exhibit how the above singularity is accomplished.

An elementary excitation of infinite lifetime is certainly an over-idealisation of the real phenomena since there is always interaction between different excitations, but for low temperature and small $\boldsymbol{k}$ it is known to be an extremely good approximation (the Landau picture of low-lying elementary excitations). On the other hand, it can be shown rigorously that an elementary excitation of finite lifetime leads to exactly the same results.

The finite lifetime of the Goldstone particle is expressed by an excitation branch with a certain width, but which is still peaked along the idealised energy–momentum curve $\sigma(\boldsymbol{k})$. Equation (5) indicates that the branch has to shrink to an exact $\delta(\omega)$ for $\boldsymbol{k} = \boldsymbol{0}$; that is, the lifetime goes to infinity for $\boldsymbol{k} \to \boldsymbol{0}$. Furthermore, the excitation exhibits a particle behaviour if the width with respect to ω of the peak as function of $\boldsymbol{k}$ is much smaller than the energy $\sigma(\boldsymbol{k})$ itself. Then a fairly general 'ansatz' is defined as follows.

Let $\phi(s)$ be a smooth function of compact support normalised to $\int \phi(s)\, \mathrm{d}s = 1$. $\sigma(\boldsymbol{k}), \chi(\boldsymbol{k})$ are continuous and, except at $\boldsymbol{k} = \boldsymbol{0}$, can be differentiated sufficiently often so that $\sigma(\boldsymbol{0}) = \chi(\boldsymbol{0}) = 0$, $\sigma(\boldsymbol{k}), \chi(\boldsymbol{k}) > 0$ for $\boldsymbol{k} \neq \boldsymbol{0}$ with $\lim_{\boldsymbol{k}\to 0} \chi(\boldsymbol{k})\sigma(\boldsymbol{k})^{-1} = 0$. We will prove that

$$J_s(\boldsymbol{k}, \omega) := J(\boldsymbol{k})\chi(\boldsymbol{k})^{-1}\phi\left(\frac{\omega - \sigma(\boldsymbol{k})}{\chi(\boldsymbol{k})}\right) \tag{10}$$

leads to SSB under the same conditions on $J(\boldsymbol{k})$ as in theorem 2.

We note that $\chi(\boldsymbol{k})^{-1}\phi[(\omega - \sigma(\boldsymbol{k}))/\chi(\boldsymbol{k})]$ is a δ sequence in ω with $\{\boldsymbol{k}\}$ as the index set. $\chi(\boldsymbol{k})$ is a measure for the width of the excitation branch and the peak is along the curve $\omega = \sigma(\boldsymbol{k})$ (with supremum $\phi(s)$ in $s = 0$).

Theorem 3. An excitation of the shape described in (10) leads to SSB if $J(\boldsymbol{k})$ is of the form $J(\boldsymbol{k}) = c(\boldsymbol{k})\sigma(\boldsymbol{k})^{-1}$ with $c(\boldsymbol{k})$ bounded in $\boldsymbol{k} = \boldsymbol{0}$ and $c(\boldsymbol{0}) \neq 0$.

Proof.

$$\lim_R \int \tilde{f}_R(\boldsymbol{k})(1-\mathrm{e}^{-\beta\omega})\,\mathrm{e}^{\mathrm{i}\omega t} J(\boldsymbol{k})\chi(\boldsymbol{k})^{-1}\phi\left(\frac{\omega-\sigma(\boldsymbol{k})}{\chi(\boldsymbol{k})}\right)\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}$$

$$=\lim_R \int \tilde{f}_R(\boldsymbol{k})\{1-\exp[-\beta(\chi(\boldsymbol{k})\omega'+\sigma(\boldsymbol{k}))]\}$$

$$\times\exp[\mathrm{i}(\chi(\boldsymbol{k})\omega'+\sigma(\boldsymbol{k}))t]J(\boldsymbol{k})\phi(\omega')\,\mathrm{d}\omega'\,\mathrm{d}\boldsymbol{k}.$$

Expanding $\exp[-\beta(\chi(\boldsymbol{k})\omega'+\sigma(\boldsymbol{k}))]$ and interchanging summation and integration, which is allowed because of the compact support of $\phi(\omega')$, we obtain

$$\beta\left(\lim_R \int \tilde{f}_R(\boldsymbol{k})J(\boldsymbol{k})\sigma(\boldsymbol{k})\exp[\mathrm{i}(\chi(\boldsymbol{k})\omega'+\sigma(\boldsymbol{k}))t]\phi(\omega')\,\mathrm{d}\omega'\,\mathrm{d}\boldsymbol{k}\right.$$

$$\left.+\lim_R \int \tilde{f}_R(\boldsymbol{k})J(\boldsymbol{k})\chi(\boldsymbol{k})\omega'\exp[\mathrm{i}(\chi(\boldsymbol{k})\omega'+\sigma(\boldsymbol{k}))t]\phi(\omega')\,\mathrm{d}\omega'\,\mathrm{d}\boldsymbol{k}\right).$$

The higher terms in the expansion do not contribute in the limit $R\to\infty$ since the higher powers in $\sigma(\boldsymbol{k})$ and $\chi(\boldsymbol{k})$ mean that they become zero $(\sigma(\boldsymbol{0})=\chi(\boldsymbol{0})=0)$. With $\int\phi(\omega')\,\mathrm{d}\omega'=1$ and $\chi(\boldsymbol{k})=\mathrm{o}(\sigma(\boldsymbol{k}))$ the above limit equals $\lim_R\int\tilde{f}_R(\boldsymbol{k})J(\boldsymbol{k})\sigma(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}$. The limit is finite and different from zero only if $J(\boldsymbol{k})=c(\boldsymbol{k})\sigma(\boldsymbol{k})^{-1}$.

We note that to adjust the general results to the models discussed in the appendix we can also write $J(\boldsymbol{k})=c'(\boldsymbol{k})(\mathrm{e}^{\beta\omega}-1)$ simply by a redefinition of $c(\boldsymbol{k})$.

Summing up what we actually have proved in this section, we can state the following. Theorem 1 is completely general and does not rely on any model assumption. Theorems 2 and 3 show that as well as a singular continuous contribution in the spectral measure, an absolutely continuous term can occur in connection with SSB. Their interpretations in physical terms are as a Goldstone mode of infinite or finite lifetime, respectively.

Some additional remarks seem to be in order concerning the physical interpretation of the above results. The excitation of theorem 2 is very reminiscent of a free Bose gas with $\sigma(\boldsymbol{k})$ as the energy of the particle. Apart from the function $c'(\boldsymbol{k})$, the spectral weight is $(\mathrm{e}^{\beta\sigma(\boldsymbol{k})}-1)^{-1}$, which is exactly the mean occupation of states of momentum $\boldsymbol{k}$ with the chemical potential vanishing, $J(\boldsymbol{k},\omega)=\int\mathrm{e}^{\mathrm{i}\omega t}\langle a^+(\boldsymbol{k},t)a(\boldsymbol{k},0)\rangle\,\mathrm{d}t=\bar{n}_{\boldsymbol{k}}\delta(\omega-\sigma(\boldsymbol{k}))$. We see that the singularity of $J(\boldsymbol{k})$ can be interpreted as the vanishing of the chemical potential μ, which can occur for two reasons.

(i) Bose–Einstein condensation of particles.

(ii) The emergence of a collective excitation, for example, magnons (see the appendix). In this case there is no particle conservation and the mean number of excitations is determined by the macroscopic variables and is therefore not an independent variable; hence $\mu=0$.

3. Cluster properties of correlation functions with respect to time and space–time

In this section we will show that the presence of a Goldstone mode alters the decay of correlations in a significant manner. This is already known for the space coordinates with the times in the correlation functions being equal. We will investigate the dynamical behaviour of correlations and give precise results concerning the decay laws in time and different space–time directions. Furthermore, we show that these properties depend sensitively on the type of Goldstone mode.

First, we state a general result which shows that the space–time cluster properties of $\langle (q(\boldsymbol{x}, t)A)\rangle^{\mathrm{T}}$ are affected in any case when a Goldstone mode shows up.

Theorem 4. If a system exhibits SSB there exists an element $A \in \mathscr{A}$ such that $\langle (q(\boldsymbol{x}, t)A)\rangle^{\mathrm{T}}$ is not absolutely integrable with respect to $(\boldsymbol{x}, t)$, which implies poor cluster properties.

Proof. The proof is very simple since almost all the work has already been done in the proof of theorem 1. In particular, part (iii) shows rigorously that $J(\boldsymbol{k}, \omega)$ is not a continuous bounded function in $(\boldsymbol{0}, 0)$. Hence the FT $\langle q(\boldsymbol{x}, t)A\rangle^{\mathrm{T}}$ cannot be a L^1 function with respect to $(\boldsymbol{x}, t)$ since the FT of a L^1 function is absolutely continuous (in the sense of functions). On the other hand, $\langle q(\boldsymbol{x}, t)A\rangle^{\mathrm{T}}$ is a smooth function without singularities, which implies that the only possibility is poor cluster behaviour for $(\boldsymbol{x}, t) \to \infty$.

While theorem 4 convinces us that there is long-range correlation in any case, it says nothing about the space–time sectors in which the decay is weakest. To get more detailed information we have to rely on theorems 2 and 3. We restrict ourselves to the ansatz of theorem 2, that is, to a sharp Goldstone excitation. On physical grounds we expect that this idealisation already exhibits the general feature of the cluster behaviour since only an infinitesimal neighbourhood of $(\boldsymbol{0}, 0)$ turns out to be relevant, and it was also shown that in any case a smeared excitation branch has to shrink to $c\delta(\omega)$ for $\boldsymbol{k} \to 0$; hence in an infinitesimal neighbourhood of $(\boldsymbol{0}, 0)$ one should be able to treat the Goldstone particle as infinitely long-lived.

For reasons of simplicity, we will deal only with two types of Goldstone modes, namely (i) $\sigma(\boldsymbol{k}) = ck^2$ and (ii) $\sigma(\boldsymbol{k}) = ck$, $k = |\boldsymbol{k}|$, which are expected to be the most important ones and which represent phonon- and magnon-like excitations respectively. Furthermore, we take only the singular part $J_{\mathrm{s}}(\boldsymbol{k}, \omega)$ of $J(\boldsymbol{k}, \omega)$ into account. The normal part $J_{\mathrm{n}}(\boldsymbol{k}, \omega)$ is expected not to affect the cluster properties.

(i) *The case* $\sigma(\boldsymbol{k}) = ck^2$. With (8) we obtain

$$\int \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega t} J_{\mathrm{s}}(\boldsymbol{k}, \omega)\, \mathrm{d}\boldsymbol{k}\, \mathrm{d}\omega = \int \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}(\mathrm{e}^{-\mathrm{i}ck^2 t} J(\boldsymbol{k}) - \mathrm{e}^{-\beta ck^2}\, \mathrm{e}^{\mathrm{i}ck^2 t} J(-\boldsymbol{k}))\, \mathrm{d}\boldsymbol{k}.$$

The method of stationary phase (see, e.g., Haag and Trych–Pohlmeyer 1977) yields for $\boldsymbol{x} \neq \boldsymbol{0}$

$$\int \exp\left[\mathrm{i}\left(\boldsymbol{u} + \frac{\boldsymbol{x}}{t}\right)\boldsymbol{x}\right] \exp\left[-\mathrm{i}c\left(\boldsymbol{u} + \frac{\boldsymbol{x}}{t}\right)^2 t\right] J\left(\boldsymbol{u} + \frac{\boldsymbol{x}}{t}\right) \mathrm{d}\boldsymbol{u}.$$

For $\boldsymbol{x}/t$ fixed and $t \to \infty$ the leading term behaves approximately as $t^{-3/2} J(\boldsymbol{x}/t)$; hence along every straight line $\boldsymbol{x}/t = \text{constant} \neq \boldsymbol{0}$, $\langle q(\boldsymbol{x}, t)A\rangle^{\mathrm{T}} := \omega(q(\boldsymbol{x}, t)A)^{\mathrm{T}}$ is absolutely integrable.

The above method becomes problematical for $\boldsymbol{x}$ fixed and $t \to \infty$ (we will deal without restriction with the case $\boldsymbol{x} = \boldsymbol{0}$, $t \to \infty$) since then $J(\boldsymbol{x}/t)$ becomes singular.

We therefore have to modify our approach slightly. Since the poor cluster properties are caused by the singularity of $J(\boldsymbol{k})$ in $\boldsymbol{k} = \boldsymbol{0}$ we make the following decomposition of the identity $1 =: f_\epsilon(|\boldsymbol{k}|) + (1 - f_\epsilon(|\boldsymbol{k}|))$ with f_ϵ a $\mathscr{C}_{00}$ function (i.e. a function which can be differentiated infinitely often) in $\boldsymbol{k}$ of shape

$$f_\epsilon = \begin{cases} 1 & |\boldsymbol{k}| \leq \epsilon \\ 0 & |\boldsymbol{k}| \geq 2\epsilon. \end{cases}$$

Then we obtain

$$\int e^{ik\cdot x-i\omega t}J_s(k,\omega)\,dk\,d\omega=\int e^{ik\cdot x-i\omega t}f_\epsilon(k)J_s(k,\omega)\,dk\,d\omega+\ldots.$$

The second term is expected to have good cluster properties because the integrand is zero in a neighbourhood U_ϵ of $k=0$ and $(1-f_\epsilon)$ is a smooth function.

Furthermore, since f_ϵ is smooth we have not introduced new artificial poor cluster properties by this splitting. So in the following we have only to deal with the term $\int e^{ik\cdot x-i\omega t}f_\epsilon(k)J_s(k,\omega)\,dk\,d\omega$.

Since $c(k)$ in $J(k)=k^{-2}c(k)$ was assumed to be continuous in $k=0$ we can write for $x=0$

$$\begin{aligned}\int e^{-i\omega t}f_\epsilon(k)J(k,\omega)\,dk\,d\omega\\ =\int f_\epsilon(k)\Big(e^{-ick^2t}\frac{1}{k^2}[c(0)-(c(0)-c(k))]-(1-\beta ck^2t\ldots)\\ \times e^{ick^2t}[c(0)-(c(0)-c(-k))]\frac{1}{k^2}\Big)\,dk.\end{aligned}\tag{11}$$

For ϵ sufficiently small the leading term yields

$$\text{constant}\times\int f_\epsilon(k)\frac{1}{k^2}c(0)\sin ck^2t\,dk=\text{constant}\times(ct)^{-1/2}\int_0^\infty f_\epsilon\Big(\frac{k'}{\sqrt{ct}}\Big)\sin k'^2\,dk'.\tag{12}$$

The integral is bounded for $t\to\infty$ and does not converge towards zero (remember the properties of f_ϵ). So we get a decay $\sim t^{-1/2}$. We obtain the following result.

When the Goldstone mode is of the type $\sigma(k)=ck^2$, the singular contribution $J_s(k,\omega)$, connected with SSB, yields the following cluster properties for $\langle q(x,t)A\rangle^T$.

(i) $\sim t^{-1/2}$ for x fixed and $t\to\infty$;

(ii) $\sim t^{-3/2}$ for a straight line $x/t=\text{constant}\neq 0$.

We can give an interesting physical interpretation of these results. We can relate the velocity of the Goldstone modes roughly with $d\sigma(k)/dk$. Then we see that for $k\to 0$ the velocity also approaches zero. In other words, the long wavelength Goldstone excitations, excited by A, stay for a relatively long time in the vicinity of the support of the operation A. So it is plausible that we get a bad decrease in the time t alone.

(ii) *The case* $\sigma(k)=ck$. Here we have two relevant cases.

For $x=0$, $t\to\infty$ we get by the same method

$$\int f_\epsilon(k)\Big(e^{-ickt}\frac{1}{k}c(0)-e^{-\beta ck}\frac{1}{k}e^{ickt}\Big)\,dk.\tag{13}$$

The leading term yields

$$\begin{aligned}&\text{constant}\times\int_0^\infty f_\epsilon(k)k\sin ckt\,dk\\ &\quad=\text{constant}\times\frac{d}{d(ct)}\int_0^\infty f_\epsilon(k)\cos ckt\,dk\\ &\quad=\tfrac{1}{2}\,\text{constant}\times\frac{d}{d(ct)}\int_{-\infty}^{+\infty}\hat{f}_\epsilon(k)\cos ckt\,dk\end{aligned}$$

with $\hat{f}_\epsilon$ the symmetric extension of f_ϵ which is also an element of $\mathscr{C}_{00}$ and of compact support. Hence the above expression falls off faster than every inverse power of t.

The second interesting case is the decay along a straight line $\boldsymbol{x}/ct$ = constant, that is $(\boldsymbol{x}, t) = (ct\hat{\boldsymbol{e}}, t)$ with $\hat{\boldsymbol{e}}$ an arbitrary but fixed unit vector. Then one obtains

$$\int \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\hat{\boldsymbol{e}}ct} f_\epsilon(k)\left(\mathrm{e}^{-\mathrm{i}ctk}\frac{1}{k} - \mathrm{e}^{-\beta ck}\frac{1}{k}\,\mathrm{e}^{\mathrm{i}ctk}\right)\mathrm{d}\boldsymbol{k}. \tag{14}$$

The leading term yields

$$\frac{\text{constant}}{ct}\int_0^\infty f_\epsilon(k)(\sin ctk)^2\,\mathrm{d}k = \frac{\text{constant}}{(ct)^2}\int_0^\infty f_\epsilon\left(\frac{k'}{ct}\right)(\sin k')^2\,\mathrm{d}k'. \tag{15}$$

Now the integral increases approximately with t since it is roughly equal to $\int_0^{\epsilon ct}(\sin k')^2\,\mathrm{d}k'$. So one gets a decrease which goes approximately as t^{-1}.

Hence a Goldstone mode with $\sigma(k) = ck$ gives the following cluster properties.

(i) Faster than every inverse power of t for x fixed and $t \to \infty$.

(ii) Proportional to t^{-1} for every straight line $(\boldsymbol{x}, t) = (ct\hat{\boldsymbol{e}}, t)$.

The physical interpretation is also transparent. Even for $k \to 0$ the velocity $\mathrm{d}\sigma(k)/\mathrm{d}k$ of the Goldstone excitations stays finite. The phonon-like excitations travel away from the support of A into all space directions with the constant velocity c. So we get poor cluster properties in every direction $(\boldsymbol{x}, t) = (ct\hat{\boldsymbol{e}}, t)$ with $\hat{\boldsymbol{e}}$ arbitrary.

4. Examples

It seems worthwhile to discuss the consequences of the general results derived above in typical physical systems.

Let us take, for example, the Heisenberg ferromagnet with spontaneous magnetisation pointing in the z direction. $S^{(x),(y),(z)}$ are the Pauli spin matrices with $[S_i^{(x)}, S_j^{(y)}] = \mathrm{i}\delta_{ij}S_i^{(z)}$, where $\boldsymbol{i}, \boldsymbol{j}$ denote the sites on the lattice $\mathbb{Z}^3$.

Hence we can make the following identifications:

$$q(\boldsymbol{x}, t) \mathrel{\hat{=}} S^{(x)}(\boldsymbol{i}, t) \qquad A \mathrel{\hat{=}} S^{(y)}(\boldsymbol{0}, 0) \qquad Q_R(t) = \sum_{|\boldsymbol{i}|\leqslant R} S^{(x)}(\boldsymbol{i}, t).$$

We get

$$\lim_{R\to\infty}\left\langle\left[\sum_{|\boldsymbol{i}|\leqslant R} S^{(x)}(\boldsymbol{i}, t), S^{(y)}(\boldsymbol{0}, 0)\right]\right\rangle = \mathrm{i}\langle S^{(z)}(\boldsymbol{0}, 0)\rangle \neq 0.$$

The Goldstone particles are the well known ferromagnetic magnons with dispersion law $\omega \sim |\boldsymbol{k}|^2$ for $\boldsymbol{k} \to \boldsymbol{0}$.

So, applying our general results, we arrive at the statements:

(i) $\langle S^{(x)}(\boldsymbol{i}, t) S^{(y)}(\boldsymbol{0}, 0)\rangle^{\mathrm{T}} \sim t^{-3/2}$ for $\boldsymbol{i}/t$ = constant;

(ii) $\langle S^{(x)}(\boldsymbol{0}, t) S^{(y)}(\boldsymbol{0}, 0)\rangle^{\mathrm{T}} \sim t^{-1/2}$.

Other interesting examples are He II and Bose–Einstein condensation.

The simplest model exhibiting a phase transition is the free Bose gas. The Goldstone particles are the Bose particles themselves; this is most easily seen by computing the commutator $\lim_{R\to\infty}\langle[\int \psi^+\psi(\boldsymbol{x})f_R(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},\, \psi^+(\boldsymbol{y})+\psi(\boldsymbol{y})]\rangle$. We suppress a more detailed discussion since this model is frequently discussed in the literature. The dispersion law of the Goldstone mode is $\omega(k)=ck^2$ and we can draw the same conclusions as above.

When a small interaction is switched on a radical change appears in the excitation spectrum. A phonon-type excitation branch emerges, which at least in the Landau model seems to be the only relevant collective excitation at low temperatures. Hence the only Goldstone excitation available is of phonon type with $\omega(k)=ck$ for $k\to 0$ and appears in the intermediate states between the density $\rho(\boldsymbol{x},t)=\psi^+\psi(\boldsymbol{x},t)$ and $\psi^+(\boldsymbol{y})+\psi(\boldsymbol{y})$. Therefore section (ii) of our general analysis does apply and we get a decay of the correlation between $\rho(\boldsymbol{x},t)$ and $\psi^+(y,0)+\psi(y,0)$ as stated in part (ii) above.

5. Summary

Combining the results we derived in the preceding sections we arrive at the conclusion that whenever a phase transition shows up in a system and is accompanied by a symmetry breaking with Goldstone particles with a sufficiently long lifetime so that our approximation works, cluster properties are affected in not only the space direction but also in time and space–time.

It is known (see, e.g., the papers by Wagner (1966) and Hohenberg (1966)) that the static cluster properties of correlation functions become poor mainly because of the spectrum of the fluctuations of certain quantities, while the exact dispersion law of the Goldstone mode seems not to be of great importance. On the other hand, the dynamical behaviour of correlations reflects in a sensitive way the character of the Goldstone mode. This comes from the fact that the static properties feel whether there is a phase transition at all only through the large fluctuations, whereas the time development of certain correlations is strongly affected by the detailed structure of the Goldstone excitation. Needless to say, these dynamical decay properties have observable effects, for example in the response functions and susceptibilities of the system.

The influence of our results on susceptibilities and response functions, perhaps physically more interesting but on the other hand more involved, together with the dependence on the space dimension will be discussed in a forthcoming paper. Furthermore, we showed in Requardt (1980) that the stability and instability of the system under extended and boundary perturbations respectively are closely connected with these poor space–time cluster properties.

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Appendix

In this appendix we will demonstrate in detail by means of two well known examples how the singularities in the two-point functions emerge and that our ansatz in theorem 2 seems to be in good agreement with the known facts.

A.1. Magnon picture of ferromagnetism

This approximative model can be found in any good textbook on solid state physics (see, e.g., Ziman 1964). One starts with the usual Hamiltonian

$$H = J \sum_{i,k} s_i \cdot s_k$$

where $\{s_i\}$ is the usual three-vector of Pauli matrices; the interaction is assumed to be of nearest-neighbour type. Introducing new operators and neglecting a fourth-order term we arrive at

$$\hat{H} = \sum_q E_q a_q^+ \cdot a_q$$

with a^+ and a the magnon creation and annihilation operators, respectively, of energy E_q, momentum $\boldsymbol{q}$ ($E_q \sim q^2$ for small $\boldsymbol{q}$). Since the magnons obey Bose–Einstein statistics one has the well known formula for the average occupation $\bar{n}_q$ of each mode:

$$\bar{n}_q = \frac{1}{e^{\beta\hbar E_q} - 1}.$$

In connection with SSB we have to calculate

$$\left\langle \left[\sum_i s_i^{(x)}, s_0^{(y)} \right] \right\rangle = \langle s_0^{(z)} \rangle = m \neq 0.$$

The spectral intensity $J(\boldsymbol{k}, \omega)$ examined in this paper simply counts the number of magnons:

$$\begin{aligned} J(\boldsymbol{k}, \omega) &= m \int \langle a_k^+(t) a_k(0) \rangle \, e^{i\omega t} \, dt \\ &= m \int \langle a_k^+(0) a_k(0) \rangle \, e^{i(\omega - E_k)t} \, dt \\ &= m \frac{1}{e^{\hbar\beta E_k} - 1} \, \delta(\omega - E_k) \end{aligned}$$

with $a_k^+(t) = a_k^+(0) \, e^{-iE_k t}$ for free particles; hence it exhibits the same singularity we derived from general principles. The difference between $(1 - e^{-\beta E_k})^{-1}$ and $(e^{\beta E_k} - 1)^{-1}$ is not accidental. The former arises from the KMS condition, the latter from a thermodynamical averaging over $a_k^+ a_k$, but the only thing important in our context is that their singular behaviour for $\boldsymbol{k} \to 0$ is of the same order, a constant $\times E_k^{-1}$ (see after the proof to theorem 3).

A.2. Bose–Einstein condensation

The next phenomenon we will examine is Bose–Einstein condensation. Contrary to § 4, we discuss the following two-point function, which has the advantage that all quantities are observables and which also shows that autocorrelation functions exhibit these singularities.

Let $\rho(\boldsymbol{x}, t)$ denote the particle density with

$$\text{FT}\, \rho_k(t) = \int d\boldsymbol{k}_1 \, e^{ik^2 t} \, e^{-i(\boldsymbol{k}+\boldsymbol{k}_1)^2 t} a^+(\boldsymbol{k}_1) a(\boldsymbol{k} + \boldsymbol{k}_1)$$

(for technical simplicity we normalise E_k to k^2). We are interested in the density–density correlation $\langle \rho_{k'}(t)\rho_{k''}(0)\rangle$ in the presence of a condensate. Therefore we have to calculate expressions like

$$\langle a^+(\boldsymbol{k}')a(\boldsymbol{k}'+\boldsymbol{q}')a^+(\boldsymbol{k}'')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle.$$

Exploiting the commutation relations for a^+, a and remembering the special role of a_0^+, a_0 we arrive at

$$\begin{aligned}&\langle a^+(\boldsymbol{k}')a(\boldsymbol{k}'+\boldsymbol{q}')a^+(\boldsymbol{k}'')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle\\ &\quad=\langle a^+(\boldsymbol{k}')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle\delta(\boldsymbol{k}'+\boldsymbol{q}'-\boldsymbol{k}'')+\langle a^+(\boldsymbol{k}')a^+(\boldsymbol{k}'')a(\boldsymbol{k}'+\boldsymbol{q}')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle.\end{aligned}$$

The second term yields

$$\begin{aligned}&\langle a^+(\boldsymbol{k}')a(\boldsymbol{k}'+\boldsymbol{q}')\rangle\langle a^+(\boldsymbol{k}'')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle\delta(\boldsymbol{q}')\delta(\boldsymbol{q}'')\\ &\quad+\langle a^+(\boldsymbol{k}')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle\langle a^+(\boldsymbol{k}'')a(\boldsymbol{k}'+\boldsymbol{q}')\rangle\delta(\boldsymbol{k}'-\boldsymbol{k}''-\boldsymbol{q}'')\delta(\boldsymbol{k}''-\boldsymbol{k}'-\boldsymbol{q}')\\ &\quad+\langle a^+(\boldsymbol{k}')a(\boldsymbol{k}'+\boldsymbol{q}')\rangle n_0\delta(\boldsymbol{q}')\delta(\boldsymbol{k}'')\delta(\boldsymbol{k}''+\boldsymbol{q}'')\\ &\quad+\langle a^+(\boldsymbol{k}'')a(\boldsymbol{k}'+\boldsymbol{q}')\rangle n_0\delta(\boldsymbol{k}''-\boldsymbol{k}'-\boldsymbol{q}')\delta(\boldsymbol{k}'')\delta(\boldsymbol{k}'+\boldsymbol{q}')\\ &\quad+\langle a^+(\boldsymbol{k}'')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle n_0\delta(\boldsymbol{q}'')\delta(\boldsymbol{k}'')\delta(\boldsymbol{k}'+\boldsymbol{q}')+n_0^2.\end{aligned}$$

We see that we have a continuous contribution which consists of the terms without condensate n_0 and which we would call according to our paper the normal part J_n. The terms with n_0 are the relevant ones for SSB. To see their impact on $J_s(\boldsymbol{k}, t)$ we have to integrate over $\boldsymbol{k}'$, $\boldsymbol{k}''$, which yields terms like

$$\begin{aligned}&\int \langle a^+(\boldsymbol{k}')a(\boldsymbol{k}''+\boldsymbol{q}'')\rangle n_0\delta(\boldsymbol{k}'')\delta(\boldsymbol{k}'+\boldsymbol{q}')\delta(\boldsymbol{k}'-\boldsymbol{k}''-\boldsymbol{q}'')\,\mathrm{e}^{\mathrm{i}k'^2t}\,\mathrm{e}^{-\mathrm{i}(k'+q')^2t}\,\mathrm{d}\boldsymbol{k}'\,\mathrm{d}\boldsymbol{k}''\\ &\quad=n_0\langle a^+(\boldsymbol{q}'')a(\boldsymbol{q}'')\rangle\,\mathrm{e}^{\mathrm{i}q''^2t},\end{aligned}$$

which exhibits exactly the same features as in the magnon case, namely

$$J_s(\boldsymbol{k},\omega)=\text{constant}\times n_0\frac{1}{\mathrm{e}^{\hbar\beta E_k}-1}\,\delta(\omega-E_k).$$

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